Derivative Pricing
Valuation Amplification
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Putting aside all reservations about the assumptions, either explicit or inherent, regarding the development of the Black-Scholes-Merton derivative pricing model, the model is truly a remarkable accomplishment. My deepest reservation regarding its development is the assumption of price movements for the underlying financial security to exhibit geometric Brownian motion. Empirical evidence shows stock prices to exhibit a random walk in linear space. Geometric Brownian motion is an unnatural thing devised solely to simplify the calculations of financial models. It exists neither in the natural world nor in the man-made world of finance. Although use of the assumption facilitates computations, it also creates a fatal flaw within any model that adopts its use. We can accept its use within the Black-Scholes-Merton derivative pricing model if we restrict its applicability only to very small price movements, which in logarithmic space would not deviate much from the symmetry occurring in linear space. But it is the lack of commutation between the logarithmic operation and the differentiation operation that destroys the mathematical veracity of any model employing both operations. Brownian motion is a random walk only in linear space.

But the model still proves useful. It does so because market participants believe its accuracy and guide their decisions by it. For algorithmic trading its usefulness is not inhibited due to its implementation on rapid time scales for small price movements. The model's use becomes questionable when employed at time scales of several months. Restricting ourselves to the applicability of only algorithmic trading on rapid time scales, the Black-Scholes-Merton derivative pricing model becomes indispensable. Yet I am bothered by one further concern regarding the model's development.

This concern only applies to the development of the model's solution for the pricing of stock options. Specifically, I refer to the transformation of both the dependent and independent variables of the partial differential equation into the form analogous of one dimensional transient heat conduction in Cartesian coordinates. Transforming their partial differential equation into the same form as that for heat conduction in a semi-infinite slab, the authors adopted the solution to the boundary value problem as their own solution. The problem I have with this is the lack of an explicit initial condition for the price of the financial derivative over the domain of permissible prices for the underlying financial security. In the heat conduction problem the semi-infinite slab is at constant uniform temperature just prior to time zero when a heat flux enters the slab at position zero. At time zero the price of the financial derivative is not known as a function of all permissible prices for its underlying financial security. This lack of specificity can be overcome by applying Green's Theorem over an arbitrary initial condition while using the heat conduction solution as the kernel of the convolution. To remove the initial condition from the convolution integral is to assume a uniform constant distribution for the initial condition. Doing so then leads a solution composed of various combinations of the error function. The authors chose to cast their solution in terms of the cumulative normal distribution, which is not as customary as is the use of the error function.

In this exposition we shall explore the general case of the Black-Scholes-Merton derivative pricing model not restricted solely to the pricing of stock options. We begin by addressing our concern regarding the arbitrary initial condition for the boundary value problem.

1 Invert Time

The initial condition to the boundary value problem poses a conundrum. We do not know the distribution of values for the derivative of the financial security over the domain of permissible prices for the security. We cannot assume that the value of the derivative is uniform over all permissible prices for the security. Nor can we assume that its distribution of value obeys Gaussian statistics. But we do know that we want the financial derivative to expire. That is, we want any financial derivative posited upon any financial security to have a value of zero after the passage of infinite time. Knowing this we can transform an ill-formed boundary value problem into one that is properly defined by inverting time.

Let us begin this inversion by first listing the customary general form of the Black-Scholes-Merton partial differential equation:

$$\frac{\partial V}{\partial t} = rV - \frac{\sigma^2 s^2}{2}\frac{\partial^2 V}{\partial s^2} - rs\frac{\partial V}{\partial s} \tag{1}$$

The boundary value problem is defined as equation (1) accompanied with the following two boundary conditions and one initial condition.

1. $V(s=0,t) = 0$ for $t \geq 0$
2. $V(s \longrightarrow \infty, t) \longrightarrow \infty$
3. $V(s, t=0) = f(s)$ where $f(s)$ is some arbitrary function of the security price.

To avoid the unknown function $f(s)$ in the initial condition of the boundary value problem we invert time. Let $\eta = 1/t$ then using this in (1) yields:

$$\eta^2 \frac{\partial V}{\partial \eta} = \frac{\sigma^2 s^2}{2} \frac{\partial^2 V}{\partial s^2} + rs \frac{\partial V}{\partial s} - rV \tag{2}$$

Where the conditions of the boundary value problem then become:

1. $V(s = 0, \eta) = 0$ for $\eta > 0$
2. $V(s \longrightarrow \infty, \eta) \longrightarrow \infty$
3. $V(s, \eta = 0) = 0$.

We now have a fully defined, well formed, boundary value problem to solve.

2 Separate Variables

We have good reason to believe that the method of separation of variables will be successful with such benign boundary conditions. Yet at this position we ought to contemplate one particular criticism directed to the Black-Scholes-Merton pricing model. It is the knowledge that neither the risk-free interest rate, r, nor the security price variance, σ^2, will remain constant with the passage of time during the life of the derivative. This indicates that, while an understanding of the rate of decay for the value of the derivative would be helpful, we are best served to explore what can be learned from time independent solutions to the boundary value problem. Considering this helps us choose the separation eigenvalue.

We take $V(s, \eta) = T(\eta)R(s)$. Using this, we separate the independent variables employing the separation eigenvalue θ, where we do not restrict the value of θ only to positive-definite values.

$$\frac{\eta^2}{T} \frac{dT}{d\eta} = \frac{\sigma^2 s^2}{2R} \frac{d^2 R}{ds^2} + \frac{rs}{R} \frac{dR}{ds} - r = \theta \tag{3}$$

Ordinarily, one would be tempted to take the separation eigenvalue as $-\theta$. Doing so would insure an anticipated exponential decay rate. Within the spatial ordinary differential equation for $R(s)$ one would be further tempted to equate θ to the risk-free interest rate r. Doing so would then indicate that the decay rate of the value for the derivative is solely determined by the risk free interest rate. But can we be certain of this knowing that neither r nor σ are truly constant over the lifetime of the financial derivative? Consequently, let us retain the general form where θ may not equal r.

The ordinary differential equation for the time dependence of the derivative valuation is readily solved.

$$T(\eta) = T_0 \exp\big(-\frac{\theta}{\eta}\big) = T_0 \exp(-\theta t) \tag{4}$$

Where T_0 is a constant.

The ordinary differential equation for the spatial part $R(s)$ requires slightly more processing.

$$s^2 \frac{d^2R}{ds} + \frac{2r}{\sigma^2} s \frac{dR}{ds} - (\theta + r)\frac{2}{\sigma^2} R = 0 \tag{5}$$

Let $\alpha = 2r/\sigma^2$ and $\beta = 2(\theta + r)/\sigma^2$. Then equation (5) becomes

$$s^2 \frac{d^2R}{ds} + \alpha s \frac{dR}{ds} - \beta R = 0 \tag{6}$$

3 Laplace Transformation

Equation (6) is a second order ordinary differential equation with variable coefficients. To solve this equation we first transform the independent variable using $z = \ln(s)$. Thus

$$\frac{d^2R}{dz^2} - (1 + \alpha)\frac{dR}{dz} - \beta R = 0 \tag{7}$$

Equation (7) is now a second order ordinary differential equation with constant coefficients. We may now use the Laplace Transform, $R(\zeta)$, to convert it into an algebraic equation.

$$R(\zeta) = \frac{\zeta - \alpha}{\zeta^2 - (1 + \alpha)\zeta - \beta} R_0 \tag{8}$$

Where R_0 is a constant.

To invert the Laplace Transform we need to find the poles of the partial fraction within equation (8). The quadratic formula yields the roots for the denominator, x_1 and x_2 as:

1. $x_1 = \frac{1+\alpha}{2} + \sqrt{(1 + \alpha)^2 + 4\beta}$
2. $x_2 = \frac{1+\alpha}{2} - \sqrt{(1 + \alpha)^2 + 4\beta}$

Using the roots in equation (8) casts it as

$$\frac{R(\zeta)}{R_0} = \frac{\zeta}{(\zeta - x_1)(\zeta - x_2)} - \frac{\alpha}{(\zeta - x_1)(\zeta - x_2)} \tag{9}$$

This form is readily inverted as

$$\frac{R(z)}{R_0} = \frac{x_1 \exp(x_1 z) - x_2 \exp(x_2 z)}{x_1 - x_2} - \alpha \frac{\exp(x_1 z) - \exp(x_2 z)}{x_1 - x_2} \tag{10}$$

We recall that $z = \ln(s)$ so equation (10) becomes

$$\frac{R(s)}{R_0} = \frac{x_1 s^{x_1} - x_2 s^{x_2}}{x_1 - x_2} - \alpha \frac{s^{x_1} - s^{x_2}}{x_1 - x_2} \tag{11}$$

4 Amplification Function

Recombining the temporal and spatial parts of the original boundary value problem yields

$$\frac{V(s,t)}{V_0} = \frac{\exp(-\theta t)}{x_1 - x_2} \big[x_1 s^{x_1} - x_2 s^{x_2} - \alpha (s^{x_1} - s^{x_2}) \big] \tag{12}$$

Where V_0 is the product of the two constants T_0 and R_0.

The ratio V/V_0 has a time dependence that can be erased either by considering the initial time only or setting θ to zero. We like this capability because we recognize that neither the risk-free interest rate r, nor the security price variance σ^2, are truly constants over the lifetime of the financial derivative. Since we are certain that the relationship found for V/V_0 is valid at time zero, for the given explicit assumptions of the model, we eliminate the time dependence by taking $t = 0$ while retaining θ as a free parameter. We define the time independent ratio V/V_0 evaluated this way as the amplification function A.

$$A(r, \sigma) = \frac{\big[x_1 s^{x_1} - x_2 s^{x_2} - \alpha (s^{x_1} - s^{x_2}) \big]}{x_1 - x_2} \tag{13}$$

The amplification is a function of the risk-free interest rate r and the security price standard deviation σ with the security price taken as a normalized parameter. At time zero, when we place an options contract bet, we set a strike price either at a discount or a premium to the current security price. That strike price is represented in normalized fashion as the parameter s in the amplification function. For example, suppose we wish to bet on an option at a ten percent premium to the current security price. If so, we then take $s = 1.1$ within the amplification function of equation (13). For a ten percent discount bet we would set $s = 0.9$ We then have a set of values for A on the ordered-pair domain of $\{r, s\}$.

The amplification represents the largest change in the valuation of the derivative, relative to the initial security price, that we can expect given a specific value of the risk-free interest

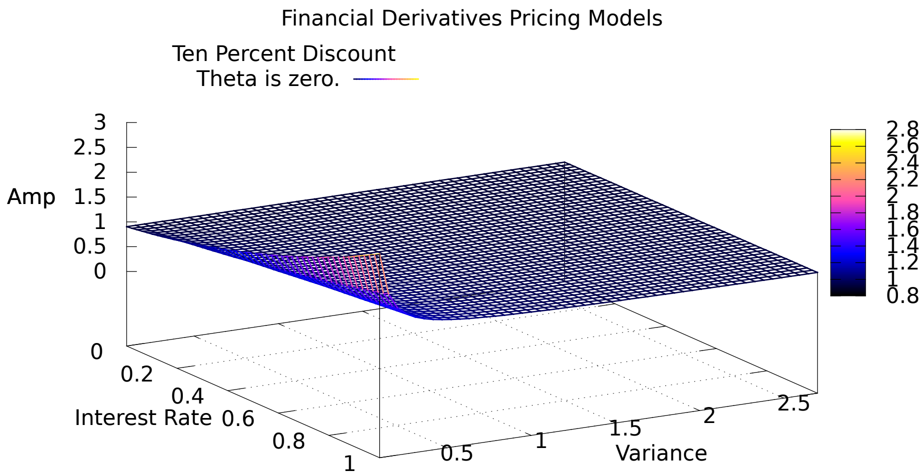

Figure 1: Ten Percent Discount Strike Price

rate and the variance of the security price when we place our bet. That is all of which we can be certain. The valuation will decay exponentially with the passage of time. But we cannot even be certain of how rapidly the decay will occur because θ remains unknown.

Strong insight of the capability for amplification can be had by examining graphs of the amplification function for various strike price ratios and values of θ. Figures 1 through 4 provide some insight. The first two graphs show the valuation amplification for a ten percent discount and a ten percent premium strike price, respectively, assuming that θ is zero. It can be seen that a larger valuation can be obtained by setting a strike price at a premium instead of at a discount. The graphs also show that the greatest effect upon amplification occurs through the interest rate and not through the price volatility. This observation is useful during low interest rate periods when one might hope that volatility would bring a strike price within range of "in the money." The last two figures show the valuation amplification for the first two cases but now with the parameter θ set to unity. With $\theta = 1$ the amplification is slightly larger, but, more importantly, the non-zero value of theta increases the effectiveness of the volatility in raising the amplification.

5 Conclusion

One can circumvent some of the criticism directed toward the Black-Scholes-Merton deriva-

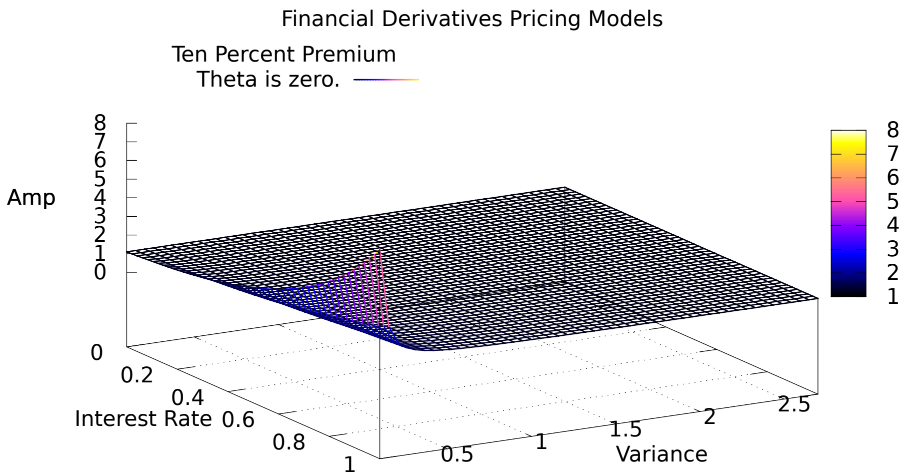

Figure 2: Ten Percent Premium Strike Price

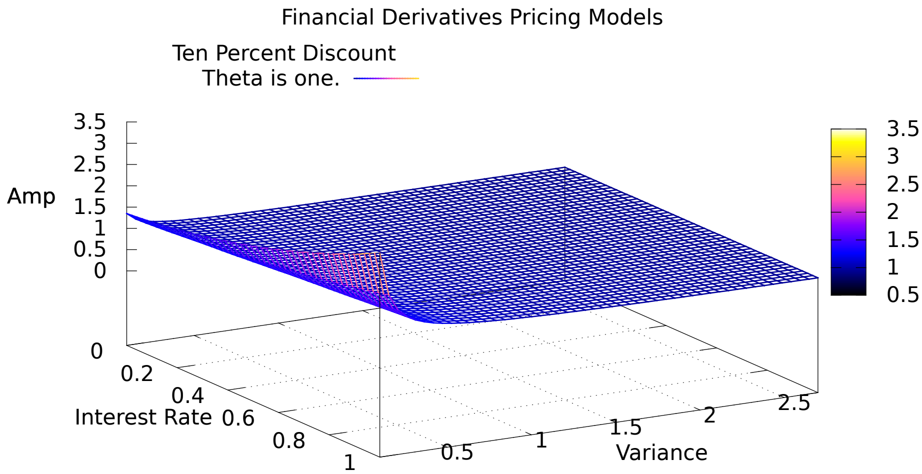

Figure 3: Ten Percent Discount with $\theta = 1$

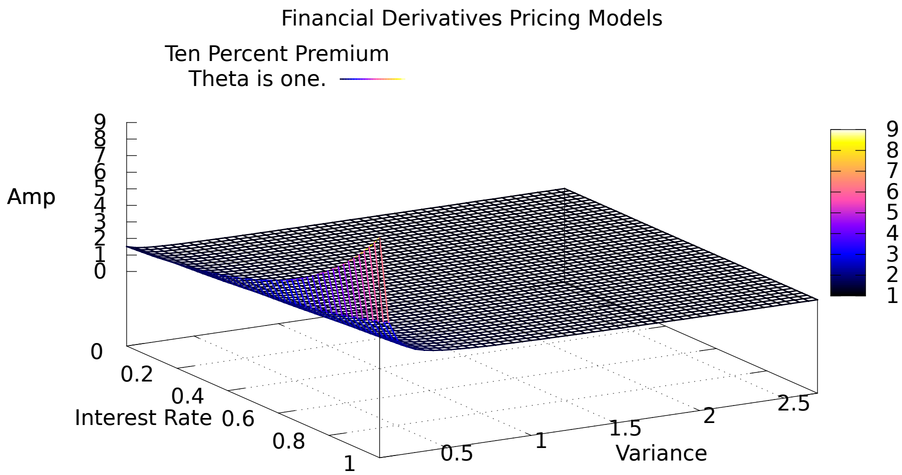

Figure 4: Ten Percent Premium with $\theta = 1$

tive pricing models by restricting analysis only to what is known as certain at the initial time when a derivative options bet is placed. Market volatility may not be capable of overcoming the handicap to valuation amplification during low interest rate periods. Although the pricing model provides no indication as to how rapidly an initial valuation will decay, faster decays do provide greater opportunity for larger amplification.