Income Reactance

Dynamic Income and the Relation Between Price and Yield

Grisafi Finance Rheology Learning Center
Steven J. Grisafi, PhD

There persists among both professionals, and the lay public, the common knowledge that the price of bonds varies inversely with respect to their interest rate. Little more is said regarding this relationship. It can exist only within a time averaged sense because, at any one time, it applies to numerous bond issues of various retirement dates. The goal of this analysis is to develop a mathematical relationship between the current price of an interest bearing financial instrument and its corresponding yield. We have in mind the relationship between the dirty price of a US Treasury bond and its current interest rate. Since such data is no longer regularly reported, all numerical calculations contained within utilize net asset value share price, and its corresponding SEC (Securities & Exchange Commission) thirty day annualized yield, for the Vanguard Long Term Treasury Fund (fund number 83). The price, ρ, and its corresponding yield, ψ are random variables whose value changes as time t progresses. No *a priori* knowledge regarding the progression of values for these random variables can be had. Recognizing that investors purchase such fixed income securities for the purpose of obtaining income, the analysis directs its attention upon dynamic income, u, which is defined empirically by the product:

$$u = \rho\psi \tag{1}$$

To be imposed upon this empirical basis is the theoretical supposition that wave phenomena are sometimes observed during the price movements of financial securities. Thus we posit upon the dependent variable u some arbitrary wave behavior over its field of independent variables ρ, ψ and also upon t. Both time varying and time invariant waves are considered. First, consider a standing wave for the income on a field composed of the price and yield:

$$\nu^2 \frac{\partial u}{\partial \psi} = u \frac{\partial u}{\partial \rho} \tag{2}$$

In equation (2) ν is a constant parameter.

The wave represented with equation (2) exhibits distortion as one progresses across its field such that a shock wave can occur. Since we have imposed this standing wave upon a field composed of both of our random variables we must check upon what conditions it could be made to possess the solution imposed upon our empirical data given by equation

(1). Substituting (1) into (2) we find that we require the constraint:

$$\nu^2 =< \psi^2 > \tag{3}$$

The square of the independent variable ψ can only be equal to the square of constant parameter ν in an average sense. Here the angle brackets indicate that the random variable ψ is to be taken as the time average over some distinct time period. For our numerical calculations we have available values for the two independent random variables over four distinct annual time periods: 2013, 2014, 2015, 2016. During these four time periods the average values for the annualized yield are 0.0298, 0.0303, 0.0253, and 0.0225, respectively.

We now consider a traveling wave. A wave which propagates with the progression of time over a field composed of only the yield is given as:

$$\tau\rho\frac{\partial u}{\partial t} + \nu^2\frac{\partial u}{\partial \psi} = 0 \tag{4}$$

The parameter τ is the period of the wave. It is a constant time scale to be determined through a curve-fit to the empirical data. The shape of this wave propagates without distortion as time progresses. However, the price enters the equation parametrically, and because ρ is a random variable, the wave form will change as time progresses in some unknown fashion.

The general solution to both the standing wave equation (2), and to the traveling wave equation (3), are both readily found using the method of characteristics for solving first order partial differential equations. Both solutions are best viewed in terms of a similarity variable. For the standing wave differential equation (2) we define the similarity variable ξ:

$$\zeta = \rho - \frac{u\psi}{\nu^2} \tag{5}$$

In terms of this similarity variable ξ the solution to the differential equation (2) is very simply: $u = H(\zeta)$, where H is some arbitrary function. However, this solution is implicitly defined because of the presence of the income, u, in the definition of the similarity variable ξ. It is this presence of u in ξ that makes possible the abrupt discontinuity of shock waves.

The general solution to the differential equation for the traveling wave of equation (4) is also best defined in terms of a similarity variable. Define the similarity variable ζ as:

$$\zeta = \psi - \frac{\nu^2 t}{\tau\rho} \tag{6}$$

With this similarity variable ζ the solution to the differential equation for the traveling wave of (4) is simply $u = F(\zeta)$, where F is some arbitrary function.

Our goal requires that we derive some relationship between the price and the yield. Although we recognize the random walk behavior of bond prices, we expect bond income to link price fluctuations with those of the yield because bond quality remains unaltered. What we desire is the rate of change of price with respect to that of the yield. But, remember, these are both independent random variables, so we would have to evaluate this ratio as:

$$\frac{\partial \rho}{\partial \psi} = \frac{\frac{\partial u}{\partial \psi}}{\frac{\partial u}{\partial \rho}} \tag{7}$$

With the differential equation for the standing wave of (2) we can immediately find this ratio through rearrangement. Doing so one finds:

$$\frac{\partial \rho}{\partial \psi} = \frac{u}{\nu^2} \tag{8}$$

When using the standing wave one would be required to evaluate ν^2 as the time averaged $< \psi^2 >$. The average income, measured in dollars, were 0.356, 0.364, 0.322, 0.294, respectively over the four time periods.

When using the traveling wave of (4) one has no means through which to evaluate the ratio of partial derivatives given by (7) because ρ appears only parametrically. To improve upon the predictions for both the standing and traveling wave equations we form a composite of the two as:

$$\tau \rho \frac{\partial u}{\partial t} = u \frac{\partial u}{\partial \rho} - \nu^2 \frac{\partial u}{\partial \psi} \tag{9}$$

The dynamic income wave represented by equation (9) propagates through a field of price and yield as time progresses. We expect the general solution to possess three undetermined constants. Inspection suggests as proposed separation of variables solutions utilizing another similarity variable φ defined as:

$$\varphi = \psi - \frac{\nu^2 t}{\tau} \tag{10}$$

The similarity variable φ removes the parametric dependence of ρ from the other similarity variable ζ. Therefore, we consider a separation of variables solution of the form $F(\varphi)G(\rho)$. Let β represent the separation eigenvalue then upon substitution into (9) one finds:

$$\frac{\nu^2}{F^2} \frac{dF}{d\varphi} = \frac{1}{\rho + 1} \frac{dG}{d\rho} = \beta \tag{11}$$

Following the usual methods one then finds as the general solution for the dynamic income:

$$u = \frac{C_0 + \beta \rho (\frac{\rho}{2} + 1)}{\beta (\frac{t}{\tau} - \frac{\psi}{\nu^2}) - C_1} \tag{12}$$

In equation (12) both C_0 and C_1 are integration constants. Along with β, ν and τ there are five undetermined constants that need to be evaluated from empirical data.

The rate of change of price with respect to yield for the income of an interest bearing financial instrument is a new quantity that we have sought. Let us represent it with the new symbol, $\mathfrak{R}$:

$$\mathfrak{R} = \frac{\partial \rho}{\partial \psi} \tag{13}$$

Evaluating the partial derivatives of equation (12) that are required for the definition of $\mathfrak{R}$ in (13) one finds:

$$\mathfrak{R} = \frac{\frac{u}{\nu^2}}{\rho + 1} \tag{14}$$

To apply equation (14) one must first evaluate the five undetermined coefficients used in the expression for the dynamic income u. Rather than attempt to fit equation (12) to time varying data, we evaluate the dynamic income as a time averaged quantity over our four distinct time periods. Inspecting the form of the right side of (12) we see that it is a ratio of polynomials on the two field variable ρ and ψ. Therefore we choose an equivalent rational function with five undetermined constants of the form:

$$u = \frac{a\rho^2 + b\rho + c}{e\psi + g} \tag{15}$$

The five unknown constants a, b, c, e and g within equation (15) are now to be evaluated from the empirical values of ρ, ψ and u as averaged quantities over the four time periods 2013, 2014, 2015 and 2016. One can now evaluate the rate of change of price with respect to yield as:

$$\mathfrak{R} = -e\frac{a\rho^2 + b\rho + c}{(2a\rho + b)(e\psi + g)} \tag{16}$$

Price is always measured in some unit of currency. Yield has the dimensions of reciprocal time. For the price and yield data of the Vanguard Long Term Treasury Fund 83, our quantity $\mathfrak{R}$ has the units of dollar-years. Let us call $\mathfrak{R}$ the income reactance.

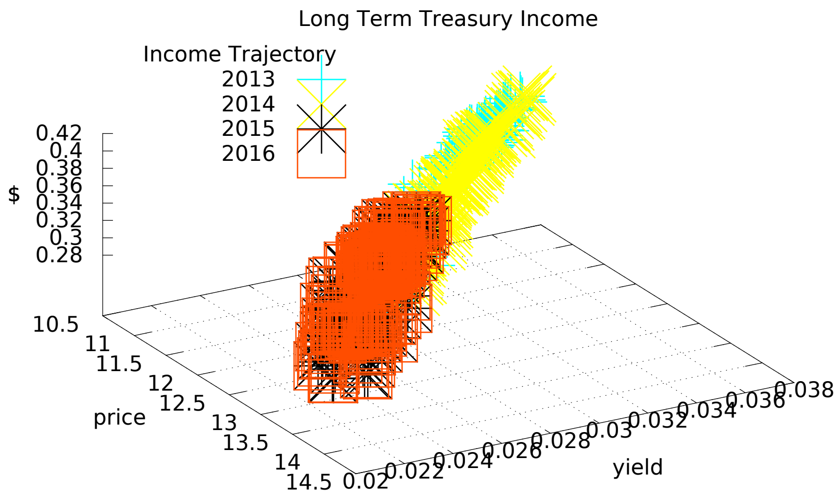

Figure 1: Income Trajectory

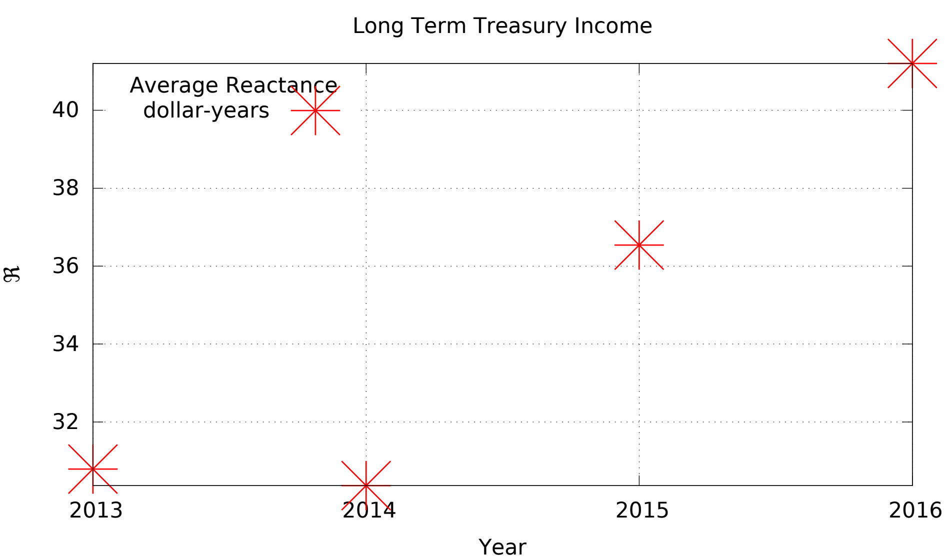

Figure 2: Average Yearly Reactance

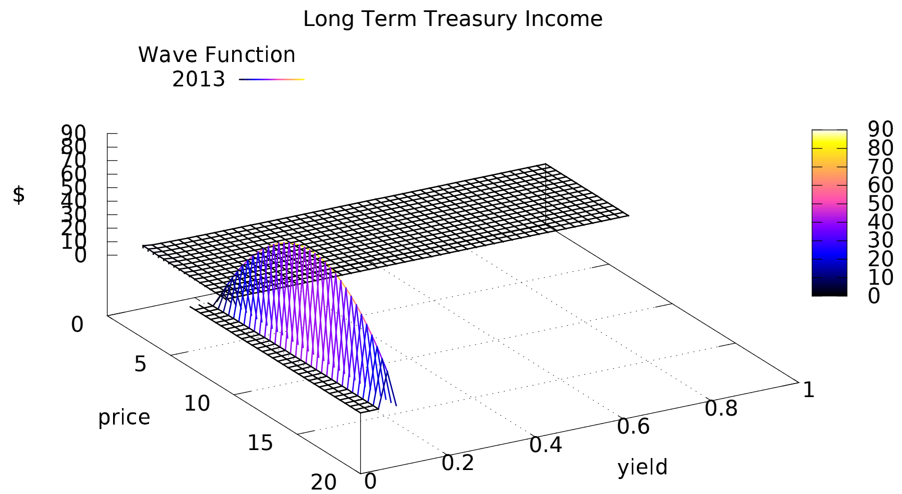

Figure 3: 2013 Wave Function

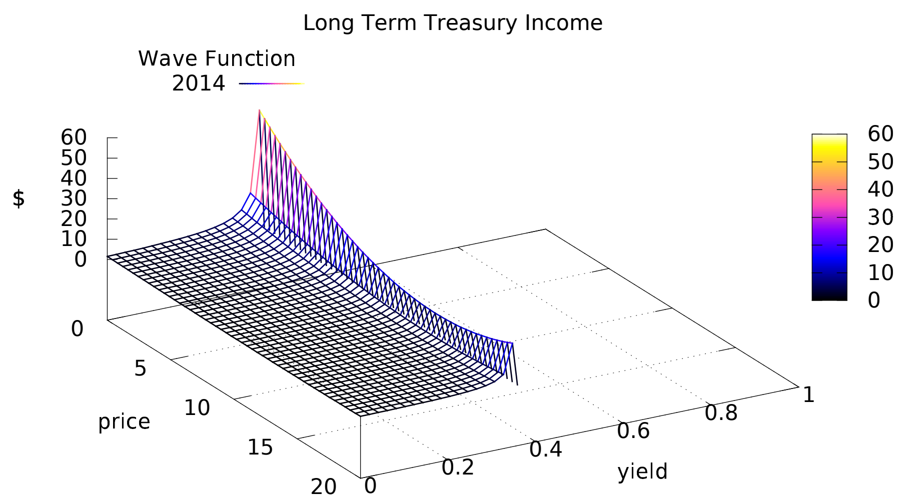

Figure 4: 2014 Wave Function

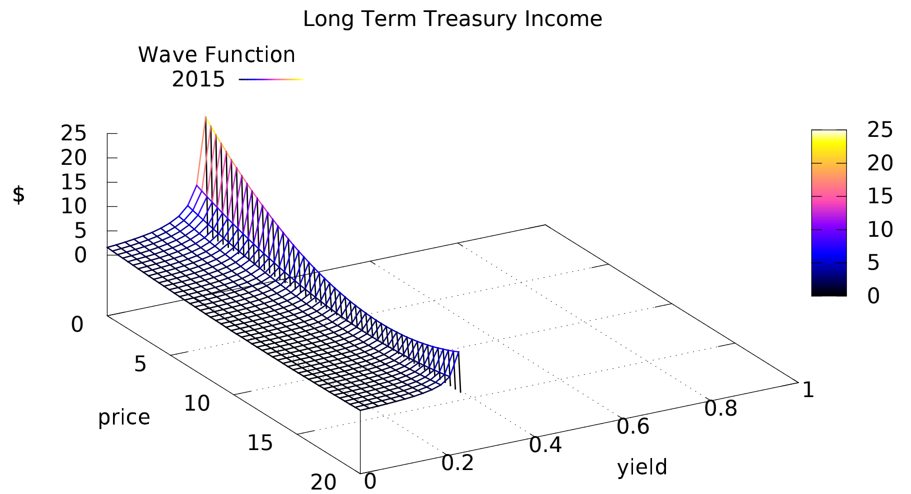

Figure 5: 2015 Wave Function

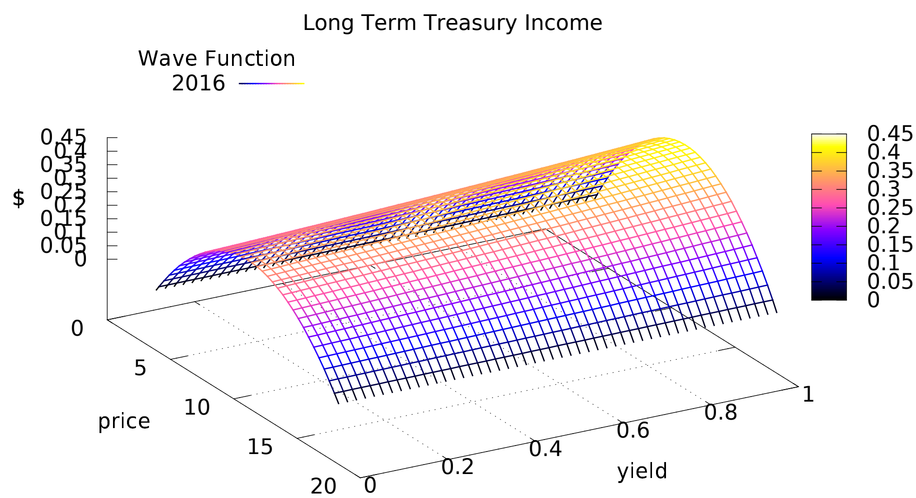

Figure 6: 2013 Wave Function

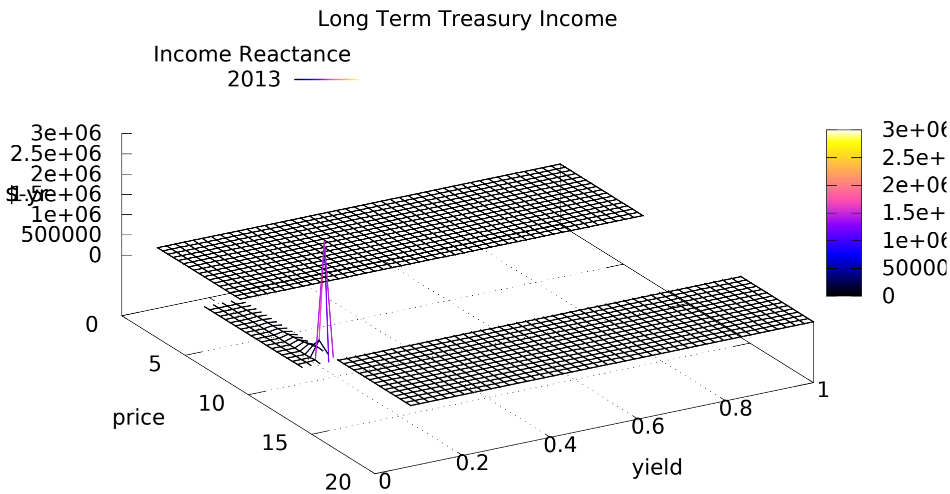

Figure 7: 2013 Income Reactance

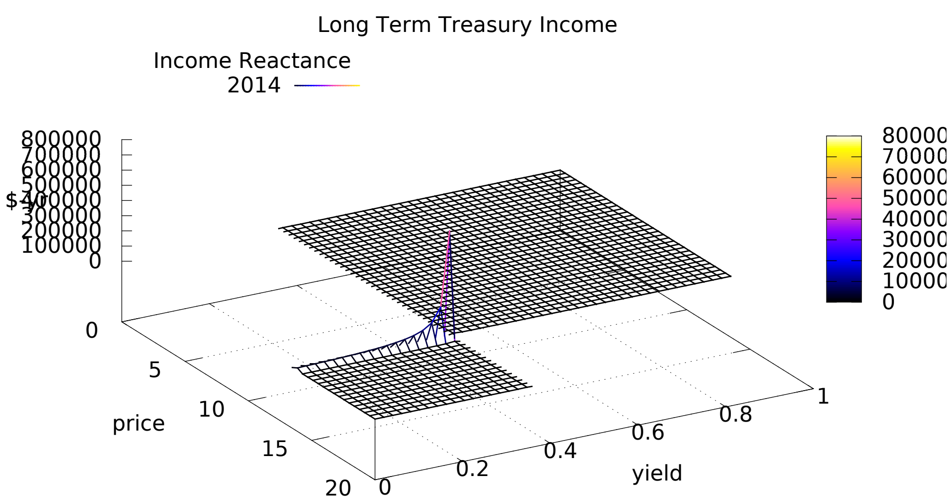

Figure 8: 2014 Income Reactance

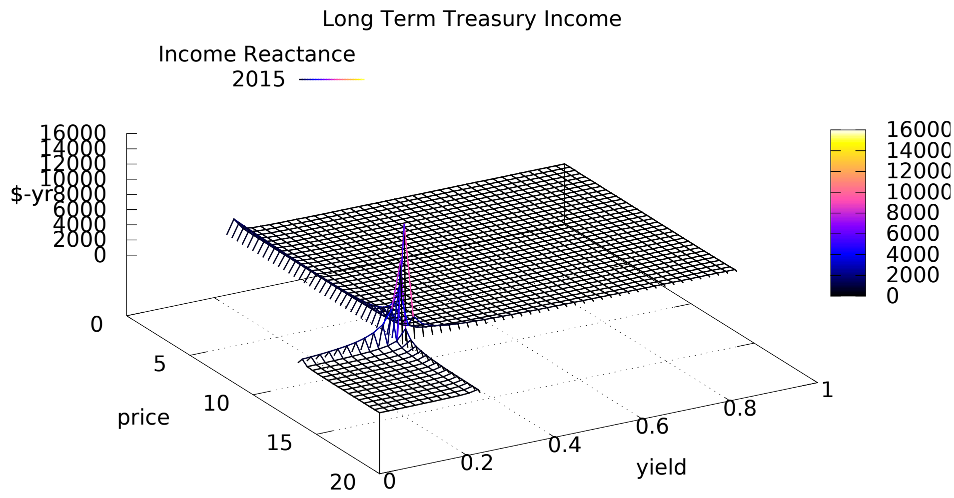

Figure 9: 2015 Income Reactance

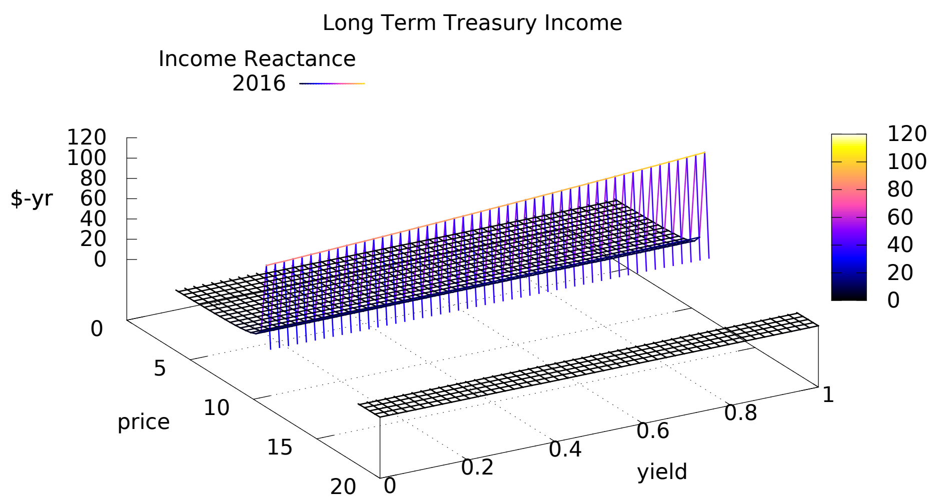

Figure 10: 2016 Income Reactance